Assignment 9.

This homework is due *Thursday*, November 5.

Collaboration is welcome. If you do collaborate, make sure to write/type your own paper and *credit your collaborators*. Your solutions should contain full proofs. Bare answers will not earn you much. Extra problems (if there are any) are due December 11.

1. Quick reminder

(B) **Lebesgue integral of bounded function.** For a bounded function on a set E of finite measure, Lebesgue integral $\int_E f$ is defined as the common value of $\sup\{\int_E \varphi \mid \varphi \le f, \varphi \text{ simple}\}$ and $\inf\{\int_E \psi \mid \psi \ge f, \psi \text{ simple}\}$, if the latter two are equal (which is guaranteed if f is measurable).
The Bounded Convergence Theorem. Let $\{f_n\}$ be a sequence of measurable functions on a set of finite measure E; let $\{f_n\}$ be uniformly bounded on E. If $\{f_n\} \to f$ pointwise on E, then $\lim_{n\to\infty} \int_E f_n = \int_E f$.

(P) **Lebesgue integral of nonnegative function.** Further, for an arbitrary *nonnegative* measurable function $f : E \to \mathbb{R} \cup +\infty$, define its Lebesgue integral by

$$\int_E f = \sup\left\{ \int_E h \;\middle|\; h \text{ bounded, measurable, of finite support and } 0 \le h \le f \text{ on } E \right\}$$

Both integrals defined above in (B) and (P) are linear, monotone and domain additive. Moreover, the following key convergence theorems hold.
Fatou's Lemma. Let $\{f_n\}$ be a sequence of nonnegative measurable functions on E. If $\{f_n\} \to f$ pointwise a.e. on E, then $\int_E f \le \liminf \int_E f_n$.
Monotone Convergence Theorem. Let $\{f_n\}$ be an increasing sequence of nonnegative measurable functions on E. If $\{f_n\} \to f$ pointwise a.e. on E, then $\int_E f = \lim \int_E f_n$.

2. Exercises

(1) (4.2.10+) The following claim was used in class in the proof of domain additivity of integral:
Lef f be a measurable function on a set E, and let A be a measurable subset of E. Then $\int_A f = \int_E \chi_A f$.
Prove it
   (a) for the definition (B) (assuming f is bounded and E is of finite measure),
   (b) for the definition (P).

(2) (4.2.13) Show that the Bounded convergence theorem fails if we drop
   (a) the assumption that the sequence $\{f_n\}$ is uniformly bounded,
   (b) the assumption $m(E) < \infty$.

(3) Let f be a *semisimple* function, i.e. a function of the form $f = \sum_{n=1}^{\infty} \lambda_n \chi_{E_n}$ for some measurable sets E_n and real numbers λ_n. Assume additionally that f is bounded and of finite support. Prove that $\int_{\mathbb{R}} f = \sum_{n=1}^{\infty} \lambda_n m(E_n)$. (*Hint:* Use the Bounded convergence theorem.)

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(4) (4.3.19) For a number $\alpha \in \mathbb{R}$, define $f(x) = x^\alpha$ for $0 < x \leq 1$ and $f(0) = 0$. Compute $\int_{[0,1]} f$. (*Hint:* In the bounded case, use connection to the Riemann integral. For the unbounded case, consider h in (P) is given by $h_M = \min\{M, x^\alpha\}$. Argue that h_M gives the largest possible value of $\int_{[0,1]} h$ for all $h \leq M$.)

(5) (4.3.18) If f is a bounded nonnegative function on a set of finite measure, both definition (B) and (P) apply to f. Show that they agree of f.

(6) ($\sim$4.3.21)
  (a) Let the function f be nonnegative and integrable over E and $\varepsilon > 0$. Show there is a simple function η on E that has finite support, $0 \leq \eta \leq f$ on E and $\int_E |f - \eta| < \varepsilon$.
  (b) Further, if E is a bounded interval, show that there is a *step* function h on E s.t. $\int_E |f - h| < \varepsilon$. (Reminder: a step function is a function of the form $\sum_{k=1}^n \lambda_k \chi_{I_k}$, where I_k are intervals.)

(7) (4.3.22+) Let $\{f_n\}$ be a sequence of nonnegative measurable functions on $\mathbb{R}$ that converges pointwise on $\mathbb{R}$ to f and f be integrable over $\mathbb{R}$. Applying the Fatou's Lemma to integrals over E and $\mathbb{R} \setminus E$, show that

$$\text{if } \int_{\mathbb{R}} f = \lim_{n\to\infty} \int_{\mathbb{R}} f_n, \text{ then } \int_E f = \lim_{n\to\infty} \int_E f_n \text{ for any measurable set } E.$$

(8) (4.3.23) Let $\{a_n\}$ be a sequence of nonnegative real numbers. Define the function f on $E = [1, \infty)$ by setting $f(x) = a_n$ if $n \leq x < n+1$. Show that $\int_E f = \sum_{n=1}^\infty a_n$ using the Monotone convergence theorem.

(9) (4.3.26) Show that the Monotone convergence theorem may not hold for decreasing sequences of functions.